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Some Fixed Point Theorems on Product of Uniform Spaces

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Abstract
Nadler found a fixed point on product of metric spaces *XxZ* for mappings on *XxZ* which are uniformly continuous and also contraction in the first variable . Fora improved Nadler's result on larger class of spaces and for larger class of mappings. Tarafdar generalized the Banach contraction principle on a complete Hausdorff uniform space. In this paper we generalize results of Nadler as well as Fora on uniform spaces. In particular, fixed point techniques have been applied in engineering, game theory, and physics. The engineering applications of fixed point theorem are to find out the optimal performance and stability of linear and nonlinear filters, image restoration and image retrieval.

I. Introduction :

A topological space X is said to have the fixed point property if every continuous function $f : X \to X$ has a fixed point.

The problem of whether the fixed point property (in short f.p.p.) is or is not necessary invariant under cartesian products is an old one (see [2] and [3] for its history). Bredon showed that the answer is negative for the category of polyhedra with the Shih condition . The f.p.p. is preserved when the maps $f: XxZ \to XxZ$ have special contraction properties . Nadler and Fora have proved results are in this direction.

A. Nadler type results

Nadler proved two main results :

A-1 Theorem : Let (X , d) be a metric space . Let $A_i : X \to X$ be a function with at least one fixed point a_i for each $i =1,2$,----------, and let $A_0 : X \to X$ be a contraction mapping with fixed point a_0. If the sequence $\{A_i\}$ converges uniformly to A_0 , then the sequence $\{a_i\}$ converges to a_0

A-2 Theorem : Let (X, d_x) be a complete metric space, let (Z, d_z) be a metric space with the f.p.p. and let f be a mapping from XxZ into XxZ. If f is uniformly continuous on XxZ and a contraction mapping in the first variable, then f has a fixed point.

We extend the class of complete metric spaces X to the class of complete Hausdorff uniform spaces and the class of metric spaces Z to the class of uniform spaces in which sequences are adequate. We prove :

A-3 Theorem : Let $(X ,\boldsymbol{u})$ be a complete Hausdorff uniform space, Z a uniform space in which sequences are adequate and which has the f.p.p. If $f : XxZ \to XxZ$ is a uniformly continuous mapping which is a contraction in the first variable , then f has a fixed point in XxZ.

Proof : Since f is contraction in the first variable, therefore for any $z \in Z$ the mapping $f_z : X \to X$ is a contraction on X. Here f_z is defined as $f_z (x) = \pi_1 f (x, z)$, where π_1 is the projection of XxZ on Z along Z.

Let $A^*(\boldsymbol{u})=\{\rho_\alpha : \alpha\in I\}$ be the augmented associated family of pseudometrics for $\boldsymbol{u}$ on X, We construct a sequence $t_n (z) = t_n$ in X as follows :

Fora fixed x_0 in X and for any $z \in Z$

$$t_0 = x_0 , t_n =\pi_1 f(t_{n-1}, z) = f_z (t_{n-1}) = f_z^n (t_0) ; n \geq 1$$

Let $\alpha \in$ I be arbitrary . If m and n are positive integers with $m > n$ then we have

$$\rho_\alpha (t_n , t_m) =\rho_\alpha (\pi_1 f(t_{n-1}, z) , \pi_1 f(t_{m-1},(z))$$

$$= \rho_\alpha\left(f_z^n(t_0), f_z^n f_z^{m-n}(t_0)\right)$$

$$\leq (\lambda_\alpha)^n \rho_\alpha\left(t_o, f_z^{m-n}(t_0)\right)$$

$$=(\lambda_\alpha)^n \rho_\alpha(t_o, t_{m-n})$$

$$\leq (\lambda_\alpha)^n[\rho_\alpha (t_0,t_1) +\rho_\alpha(t_1,t_2) +\text{-----}+\rho_\alpha(t_{m-n-1},t_{m-n})]$$

$$\leq (\lambda_\alpha)^n \rho_\alpha (t_0,t_1) [1+\lambda_\alpha +\text{----}+(\lambda_\alpha)^{m-n-1}]$$

$$< \frac{(\lambda_\alpha)^n}{1-\lambda_\alpha} \rho_\alpha (t_0, t_1)$$

$$\to 0 \text{ as, } n \to \infty$$

Above inequality implies $\{t_n\}$ is a ρ_α - Cauchy sequence (ie a Cauchy sequence in ρ_α topology). Since $\alpha \in I$ is arbitrary, $\{t_n\}$ is a ρ_α - Cauchy sequence .

Let $S_p = \{t_n : n \geq p\}$ for all positive integer p and let $B = \{S_p : p=1,2----\}$ be the filter basis. It is easy to see the filter basis B is Cauchy in the uniform space $(X, \boldsymbol{u})$ To see this we first note that the family $\{H(\alpha, \in) : \alpha \in I, \in > 0\}$ is a base for $\boldsymbol{u}$ .Now let $H \in \boldsymbol{u}$ be an arbitrary entourage. Then there exists $a\ v \in I$ and $\in > 0$) such that $H(v, \in) \subset H$. Since $\{t_n\}$ is a ρ_v - Cauchy sequence in X, there exists a positive integer p such that $\rho_v (t_n, t_m) < \in$ for $m \geq p, n \geq p$ this implied that $S_p x S_p \subset H(v, \in)$. Thus given any $H \in \boldsymbol{u}$ we can find a $S_p \in B$ such that $S_p x S_p \subset H$ .Hence B is a Cauchy filter in $(X, \boldsymbol{u})$.Since $(X, \boldsymbol{u})$ is complete and Housdorff, the Cauchy filter $B = \{S_p\}$ converges to a unique point $a \in X$ in the τ_u topology (uniform topology induced by uniformity $\boldsymbol{u}$). Thus $\tau_u \lim S_p = a$. Now since f_z is ρ_α - continuous for each $\alpha \in I$, it follows that f_z is τ_u continuous .

Hence $f_z(a) = f_z(\tau_u \lim S_p) = \tau_u \lim f_z(S_p) = \tau_u \lim S_{p+1} = a$.Thus a is a fixed point of f_z. Here a is unique fixed point of f_z as if we assume b is another fixed point of f_z such that $a \neq b$. Since $(X, \boldsymbol{u})$ is a Hausdorff space and $a \neq b$, there is an index $\beta \in I$ such that $\rho_\beta (a,b) \neq 0$. Since f_z is a contraction on X, we have

$$\rho_\beta (a,b) = \rho_\beta (f_z (a), f_z (b)) \leq \lambda_\beta \rho_\beta (a, b)$$

Which is absurd as $0 < \lambda_\beta < 1$ and $\rho_\beta (a,b) \neq 0$. Hence a is unique fixed point of f_z .

Let $F : Z \to X$ be given by $F(z) = a$ the unique fixed point of f_z . Now let $z_0 \in Z$ and let $\{z_i\}$ be a sequence of points of Z which converges to z_0. By the assumption of this theorem, the sequence $\{f_{zi}\}$ converges uniformly to f_{z0} and hence, by theorem A_1, the sequence $\{F(z_i)\}$ converges to $F(z_0)$. Therefore F is continuous on Z. Next let $G : Z \to Z$ be the continuous mapping defined by $G(z) = \pi_2 f(F(z),z)$ for each $z \in Z$, where π_2 is the projection of $X \times Z$ on Z along X. Since Z has the f.p.p. there is a point $p \in Z$ Such that $G(p) = p$. Therefore $p = G(p) = \pi_2 f(F(p),p)$. It follows that $(F(p),p)$ is a fixed point of f. This completes the proof of the theorem.

A-4 Corollary : Let $(X, \boldsymbol{u})$ be a complete Hausdorff uniform space and let Z a uniform space in which sequences are adequate and which have the f.p.p. If $f: XxZ \to XxZ$ is a mapping which is a contraction mapping in each variable separately then f has a fixed point in XxZ.

Here we note that Theorem A-2 also corollary of above Theorem A-3.

B. Fora type results :

Fora's improvements of Nadler's results are based on the observation that in Nadler's results , metric character of Z is not necessary, uniform continuity of f is too strong and contraction condition is sufficient even if it is available locally. Therefore Fora replaced X by a complete metric space, Z by any topological space, uniformly continuous f by a continuous f and f being contraction in the first variable by the condition that f is locally contraction in the first variable. We generalize Fora's result as follows:

B-1 Theorem : Let $(X, \boldsymbol{u})$ be a complete Hausdorff uniform space, Z a topological space with the f.p.p., $f: XxZ \to XxZ$ be a locally contraction mapping in the first variable . If f is continuous when the topology on X is given by any uniformly continuous pseudometric on $X \times Z$, then f has a fixed point.

Proof : Let $\{\rho_\alpha : \alpha \in I\}$ be the collection of all uniformly continuous pseudometrics on X. Let $x_0 \in X$ be fixed and for any $z \in Z$, we construct a sequence $t_n(z) = t_n$ in X as follows:

$t_0 = x_0$, $t_n = \pi_1 f(t_{n-1}, z)$; $n \geq 1$

Step –I : $\{t_n\}$ is a Cauchy sequence in $(X, \boldsymbol{u})$

Since f is locally contraction in the first variable, for each $\alpha \in I$ there exists a real number $\lambda_\alpha \in [0,1)$ such that

$\rho_\alpha (\pi_1 f(t_{n-1}, z), \pi_1 f(t_n, z)) \leq \lambda_\alpha \rho_\alpha (t_{n-1}, t_n)$

or $\quad \rho_\alpha (t_n, t_{n+1}) \leq \lambda_\alpha \rho_\alpha (t_{n-1}, t_n)$

Using triangular inequality , we find for $m>n$

$\rho_\alpha (t_n, t_m) \leq \rho_\alpha (t_n, t_{n+1}) + \rho_\alpha (t_{n+1}, t_{n+2}) + \text{-------} + \rho_\alpha (t_{m-1}, t_m)$

$$\leq \left(\lambda_\alpha^n + \lambda_\alpha^{n+1} + --- + \lambda_\alpha^{m-1}\right) \rho_\alpha (t_0, t_1)$$

$$= \frac{\lambda_\alpha^n \left(1-\lambda_\alpha^{m-n}\right)}{1-\lambda_\alpha} \rho_\alpha (t_0, t_1)$$

$$< \frac{\lambda_\alpha^n}{1-\lambda_\alpha} \rho_\alpha (t_0, t_1)$$

Since $\lambda_\alpha^n \to 0$ as $n \to \infty$, this inequality shows that $\{t_n\}$ is a ρ_α- Cauchy sequence(ie a Cauchy sequence in ρ_α-topology). Since $\alpha \in I$ is arbitrary, $\{t_n\}$ is a ρ_α- Cauchy sequence .

Let $B = \{S_p : p \in N\}$where $S_p = \{t_n : n \geq p\}$ be a Cauchy filter base in $(X, \boldsymbol{u})$. To see this we first note that the family, $\{H(\alpha,\in):\alpha\in I, \in>0\}$ is a base fo $\boldsymbol{u}$ as $A^*(\boldsymbol{u}) = \{\rho_\alpha : \alpha \in I\}$. Now let $H\in\boldsymbol{u}$ be an arbitrary entourage.Then there exist a $v\in I$ and $\in>0$ such that $H(v, \in)\subset H$. Now since $\{t_n\}$ is a ρ_α-Cauchy sequence in X, there exists a positive integer P such that $\rho_v(t_n, t_m) < \in$ for all $m\geq p$, $n \geq p$. This implies that $S_p x S_p \subset H\ (v, \in)$. Thus given any $H \in \boldsymbol{u}$ we can find a $S_p\in B$ such that $S_p x S_p \subset H$. Hence B is a Cauchy filter in $(X, \boldsymbol{u})$. Since $(X, \boldsymbol{u})$ is complete and Hausdorff, the Cauchy filter $B = \{S_p\}$ converges to a point say t_z in X.

Let mapping $g : Z\to Z$ defined as $g(z) = \pi_2 f(t_z , z)$ where π_2 is the projection of X x Z on Z along X.

Step II : $g : Z\to Z$ is continuous.

Let $z\in Z$ and U be an open set containing $g(z)$.
Then $f(t_z , z) \in XxU$. Since f is continuous at $(t_z , z)$when X is assigned the topology $\tau(\rho)$ in which $\rho \in A^*(\boldsymbol{u})$ implies $\rho =\rho_\alpha$ for some $\alpha\in I$, there exists an open set $G\subset Z$ and a real number $\in>0$ such that

$$(t_z , z) \in S(t_z , \in,\rho) \, x \, G \text{ and}$$
$$f\{S(t_z , \in, \rho) \, x \, G\subset XxU$$

Also f is locally contraction in the first variable . Therefore there exists an open set W, containing z and $\lambda\in[0,1)$ such that

$$\rho(\pi_1 f(x , v) , \pi_1 f(x_*,v)) \leq \lambda\, \rho(x , x_*)$$

for all x , $x_*\in X$ and all $v\in W$.
Since $\lambda^m \to 0$ as $m\to\infty$, we all choose $n \geq 1$ such that

$$\lambda^n < \frac{\in}{8}\left(\frac{1-\lambda}{\rho(t_0,t_1)+(\in/8)}\right) \text{and } \rho(t_z , t_m) < \frac{\in}{8}$$

for all $m \geq n$

Since $f(t_n, z) \in XxU$ and f is continuous at (t_n , z) , there exists a basic open set $U_n x V_n$ in $X x Z$ such that

$$(t_n , z) \in U_n xV_n \ , U_n \subset S(\frac{\in}{8}, t_z, \rho) ,$$
$$V_n \subset G\cap W \text{ and } f(U_n xV_n)\subset XxU.$$

Since f is continuous at (t_{n-1}, z) and $f(t_{n-1}, z)\in U_n xZ$, there exists a basic open set $U_{n-1} xV_{n-1}$ in XxZ such that

$$(t_{n-1}, z) \in U_{n-1} xV_{n-1}, U_{n-1} \subset S(\frac{\in}{8}, t_{n-1} , \rho) ,$$

$V_{n-1}\subset V_n$ and $f(U_{n-1} xV_{n-1})\subset U_n xZ$.
Continuing this way we construct sets $U_n ,U_{n-1} ,\ldots,U_0 ,V_n ,V_{n-1},\ldots,V_0$ such that, for $0\leq i \leq (n-1)$

$$(t_i , z) \in U_i xV_i \ , U_i\subset S(\frac{\in}{8}, t_i , \rho) ,$$
$$V_i \subset V_{i+1} \text{ and } f(U_i xV_i) \subset U_{i+1} x Z.$$

It remains to show that $g(V_0) \subset U$:

Let $y \in V_0$. Then from the above mention properties we have $(t'_0, y) \in U_0 xV_0$,
Where $t'_0 = x_o$. Thus $f(t'_0 , y) \in U_1xZ$ ie. , $t'_1 = \pi_1 f(t'_0 , y) \in U_1$,

consequently $\rho(t'_1 , t_1) < \frac{\in}{8}$.

Using the triangular inequality we have
$\rho(t'_0 , t'_1) =\rho(t_0 , t_1') \leq\rho(t_0 , t_1) + \rho(t_1 , t'_1) < \rho(t_0 , t_1) + \frac{\in}{8}$.

Since $f(U_1xV_1)\subset U_2xZ$ and $(t'_1 , y)\in U_1xV_1$ therefore $f(t'_1 , y)\in U_2xZ$
ie $t'_2= \pi_1 f(t'_1 , y) \in U_2$.
In this way we find the sequence $t'_n(y)= t'_n$,forwhich $t'_i= \pi_1 f(t'_{i-1}, y) \in U_i$; $i=1,2,---, n$.

Moreover, $t'_n \in U_n$ and $U_n\subset S(\frac{\in}{8}, t_z, \rho)$, therefore

$\rho(t'_n , t_z) < \frac{\in}{8}$.

Using the triangle inequality we find, for m≥n.
$\rho(t'_n , t_z) \leq \rho(t_z , t'_n) + \rho(t'_n , t'_{n+1})$ +----------+ $\rho(t'_{m-1} , t'_m)$

$$< \frac{\in}{8} +\lambda^n \rho(t'_0 , t'_1) +\text{----}+\lambda^{m-1} \rho(t'_0 , t'_1)$$
$$= \frac{\lambda^n}{1-\lambda}\rho(t'_0 , t'_1) + \frac{\in}{8}$$
$$<(\frac{\lambda^n}{1-\lambda}) (\rho(t'_0 , t'_1) + \frac{\in}{8})+ \frac{\in}{8}$$
$$< \frac{\in}{8} + \frac{\in}{8} = \frac{\in}{4}$$

If $t_y = \lim t'_n$, then the above inequality shows that $\rho(t_y , t_z) \leq \in/4$.Therefore $(t_y , y) \in S(\in, t_z ,\rho)$

and consequently $f(t_y , y) \in XxU$, ie. , $g(y) = \pi_1 f(t_y , y) \in U$. Therefore our claim is proved and

hence g is continuous.

Step –III : $\pi_1 f(t_z, z) = t_z$

If possible , let $u = \pi_1 f(t_z , z) \neq t_z$. Since the uniform space X is Hausdorff, there exists a pseudometric ρ on X such that $\rho(u , t_z)= \in>0$.

Since f is continuous on XxZ and X is assigned the topology $\tau(\rho)$, we have open sets U and V such that

$(t_z, z) \in U \times V$, $U \subset S(\frac{\in}{4}, t_z, \rho)$ and $f(U \times V) \subset S(\frac{\in}{4}, u, \rho) \times Z$.

Since lim $t_n = t_z$, there is a natural number $k \geq 1$ such that $t_n \in U$ for all $n \geq k$. Therefore

$f(t_k, z) \in S(\frac{\in}{4}, u, \rho) \times Z$, ie $t_{k+1} = \pi_1 f(t_k, z) \in S(\frac{\in}{4}, u, \rho)$.

Also $t_{k+1} \in U \subset S(\frac{\in}{4}, t_z, \rho)$.This contradicts the fact that $\rho(t_z, u) = \in$. Therefore our assumption is false and consequently we have the required conclusion.

Now as in step II of the theorem *B-1*, $g: Z \to Z$ is continuous. Since Z has the fixed point property, therefore there exist $z_0 \in Z$ such that $g(z_0) = z_0$. As in **step-III** above we have $\pi_1 f(t_{z0}, z_0) = t_{z0}$. But $z_0 = g(z_o) = \pi_2 f(t_{z0}, z_0)$. Hence $f(t_{z0}, z_0) = (t_{z0}, z_0)$ ie, (t_{z0}, z_0) is a fixed point of f. This completes the proof.

It is obvious that Theorem A-2 is a corollary to the above Theorem B-1 . We also get as a corollary to this Theorem the following result mentioned by Fora [4].

B-2 Corollary : Let (X, d) be a complete metric space , Z be a topological space with the f.p.p. and $f: X \times Z \to X \times Z$ a continuous mapping. If f is a contraction in the first variable , then f has *a* fixed point.

References

[1] **Bredon G.:** 'Some examples of fixed point property', Pacific J. Math. 38 (1971), 571-573

[2] **Brown R.F. :** 'On some old problems of fixed point theory', Rocky Mountain. J. Math., 4 (1974), 3-14.

[3] **Fadell E.R. :** 'Recent results in the fixed point theory of continuous maps,' Bull. Amer. Math. Soc., 76 (1970), 10-29.

[4] **Fora A.A. :** 'A fixed point theorem for product spaces' pacific J. Math. 99 (1982), 327-335

[5] **Husseni S.Y.** : 'The product of manifolds with f.p.p. need not have the f.p.p.,' Amer.J. 99 (1977) ,919-931.

[6] **Lee Cheng-Ming :** 'A development of contraction mapping principles on Hausdorff uniform spaces', Trans. Amer. Math. Soc. (1977), 147-159.

[7] **Nadler Jr. S.B. :** 'Sequence of contractions and fixed points', Pacific.J. Math.. 27 (1968), 579-585.

[8] **Rhoades B.E. :** 'A Comparison of various definitions of contractive mappings', Trans. Amer.Math.Soc. 226 (1977), 257-290.

[9] **Tarafdar E.:** 'An approach to fixed point theorems in uniform spaces', Trans. Amer.Math. Soc. 191 (1974), 209-225.